Time required to heat mass in constant-volume process

1 Prompt

Let there be a closed room of volume V at indoor temperature T_1. It has to be cooled down to temperature T_2. If Q_{out} amount of heat is removed from the room every second, then how much time will it take to reach the desired temperature?

Assume Q_{in} is the amount of heat input to the room every second.

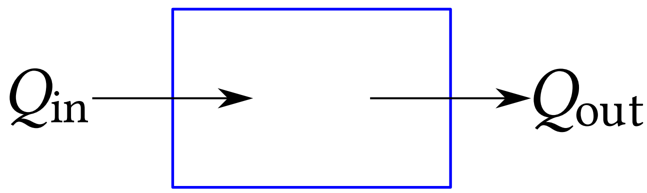

Figure 1. Heat flows into and out of the container.

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1 Solution

1.1 Energy balance

First, consider an energy balance of the system.

$$(\text{rate of energy in}) - (\text{rate of energy out}) + (\text{rate of energy generation}) = (\text{rate of energy accumulation}) \tag{1}$$

$$Q_{\text{in}} - Q_{\text{out}} + 0 = \frac{dU}{dt} \cdot m \tag{2}$$

Where:

Q_{in}	: rate of heat flow into the system	$\left[\frac{\text{J}}{\text{s}}\right]$
Q_{out}	: rate of heat flow out of the system	$\left[\frac{\text{J}}{\text{s}}\right]$
t	: time	$[\text{s}]$
U	: internal energy of the system per unit mass	$\left[\frac{\text{J}}{\text{kg}\cdot\text{s}}\right]$
m	: mass of system	$[\text{kg}]$

Solve for dU.

$$dU = \left(\frac{1}{m}\right) \cdot (Q_{\text{in}} - Q_{\text{out}}) \cdot dt \tag{3}$$

Integrate from initial to final state.

$$\int_{U=U_i}^{U=U_f} dU = \frac{1}{m} \cdot \int_{t=t_i}^{t=t_f} (Q_{\text{in}} - Q_{\text{out}})\, dt \tag{4}$$

$$U_f - U_i = \frac{1}{m} \cdot (Q_{\text{in}} - Q_{\text{out}}) \cdot (t_f - t_i) \tag{5}$$

Simplify with $\Delta U = U_f - U_i$.

$$\Delta U = \frac{1}{m} \cdot (Q_{\text{in}} - Q_{\text{out}}) \cdot (t_f - t_i) \tag{6}$$

1.2 Temperature change from heat flow.

For a mechanically reversible constant-volume process, $Q = m \cdot \Delta U$.

Take into account changes in temperature of the system mass by introducing the concept of constant-volume heat capacity. The definition of constant-volume heat capacity of a substance (provided by Introduction to Chemical Engineering, 7th Edition, by J.M. Smith) is:

$$C_V \equiv \left(\frac{dU}{dT}\right)_V \tag{7}$$

Where:

$$\begin{aligned} C_V &: \text{constant volume heat capacity} && \left[\frac{\text{J}}{\text{kg}\cdot\text{K}}\right] \\ T &: \text{temperature of the system} && [\text{K}] \\ V &: \text{specific volume of the system} && \left[\frac{\text{m}^3}{\text{kg}}\right] \end{aligned}$$

$$dU = C_V\, dT \tag{8}$$

$$\int_{U=U_i}^{U=U_f} dU = \int_{T=T_i}^{T=T_f} C_V\, dT \tag{9}$$

$$U_f - U_i = C_V \cdot (T_f - T_i) \tag{10}$$

Substitute $\Delta U = U_f - U_i$:

$$\Delta U = C_V \cdot (T_f - T_i) \tag{11}$$

1.3 Combine energy rate balance equation with energy-temperature equation

Now we have an expression (equation 11) for how the internal energy of the constant-volume system changes with temperature. We also have an expression (equation 6) for how internal energy changes according to heat flows Q_{in} and Q_{out}. These two changes in internal energy are equal so we set equation 6 and equation 11 equal to one another, yielding equation 12.

$$\Delta U = \frac{1}{m} \cdot (Q_{\text{in}} - Q_{\text{out}}) \cdot (t_f - t_i) = C_V \cdot (T_f - T_i) \tag{12}$$

Solve for $t_f - t_i$.

$$t_f = \frac{C_V \cdot m \cdot (T_f - T_i)}{Q_{\text{in}} - Q_{\text{out}}} + t_i \tag{13}$$

Now, apply the prompt's parameters.

$$\begin{aligned} T_i &= T_1 \\ T_f &= T_2 \\ t_i &= 0 \end{aligned}$$

Solve for t_f.

$$t_f = \frac{C_V \cdot m \cdot (T_2 - T_1)}{Q_{\text{in}} - Q_{\text{out}}} \tag{14}$$

1.4 Summary

Equation 14 is the amount of time required to cool mass m from T_1 to T_2, provided the constant-volume specific heat capacity C_V is known as well as the heat flows to (Q_{in}) and from (Q_{out}) the rigid container.

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